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Symmetry preserving eigenvalue embedding in finite-element model updating of vibrating structures

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Abstract

The eigenvalue embedding problem addressed in this paper is the one of reassigning a few troublesome eigenvalues of a symmetric finite-element model to some suitable chosen ones, in such a way that the updated model remains symmetric and the remaining large number of eigenvalues and eigenvectors of the original model is to remain unchanged. The problem naturally arises in stabilizing a large-scale system or combating dangerous vibrations, which can be responsible for undesired phenomena such as resonance, in large vibrating structures. A new computationally efficient and symmetry preserving method and associated theories are presented in this paper. The model is updated using low-rank symmetric updates and other computational requirements of the method include only simple operations such as matrix multiplications and solutions of low-order algebraic linear systems. These features make the method practical for large-scale applications. The results of numerical experiments on the simulated data obtained from the Boeing company and on some benchmark examples are presented to show the accuracy of the method. Computable error bounds for the updated matrices are also given by means of rigorous mathematical analysis.

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1. Introduction

Vibrating structures such as bridges, highways, buildings, automobiles, air and space crafts, etc., are very often modelled by using finite-element methods (FEMs). These methods generate structured systems of matrix second-order differential equations of the form

$$M\ddot{x} + C\dot{x} + Kx = 0, \tag{1}$$

where the coefficient matrices M, C and K are called, respectively, the mass, damping and stiffness matrices. In most applications, these matrices have very special exploitable properties such as the symmetry, positive definiteness, sparsity and others. The matrix M is often symmetric positive definite and denoted by $M>0$; and K is symmetric positive semi-definite, denoted by $K\geqslant 0$. The damping matrix C is hard to determine in practice; however, very often, for the sake of computational convenience and other practical considerations, it is assumed to be symmetric.

It is critical and very important that these properties are preserved while solving a vibration problem or updating a FEM to achieve certain design objectives.

In this paper, we will assume throughout that $M>0$, $K>0$ and $C = C^{\mathrm{T}}$.

The classical approach is to use separation of variables, accounting for a solution $x(t) = ye^{\lambda t}$ to (1), where y is a constant vector. This leads to the quadratic matrix eigenvalue problem

$$F(\lambda_k)y_k = 0, \quad k = 1, 2, \ldots, 2n,$$

where

$$F(\lambda) = \lambda^2 M + \lambda C + K \tag{2}$$

is the so-called associated quadratic matrix pencil. The quantities (λ_k, y_k), $k = 1, \ldots, 2n$ are the eigenpairs of the pencil (2).

It is well-known [1] that the dynamical behavior of a vibrating system, which can show undesired phenomena such as instability and resonance, is determined by their natural frequencies and corresponding mode shapes, that is, the eigenvalues and eigenvectors of the pencil $F(\lambda)$. It is desirable that such behaviors are altered by making minimal changes in the system and keeping the structural properties invariant, as much as possible. Realistically, while dealing with a large system, it is often found in practice that only a small number of eigenvalues are "troublesome". Thus, it makes sense to reassign to suitable locations, chosen by the designer, only these troublesome eigenvalues, while keeping the remaining large number of eigenvalues unchanged.

Such a problem in control theory is known as the *partial pole-placement problem* and feedback control is used to solve this problem. For the standard first-order state–space systems of the form $\dot{x}(t) = Ax(t) + Bu(t)$, though there exist many numerical methods for the complete pole-placement (see Ref. [2] for details), only two methods have so far been developed for the partial pole-placement problem: (i) the projection method due to Saad [3], and (ii) the Sylvester equation method by Datta and Sarkissian [4]. For a matrix second-order system, the choices are either to transform the latter to a standard first-order form and then use one of the above methods or to use the Independent Modal Space Control (IMSC) approach [1]. Both have some severe engineering and computational limitations. The first approach might require an ill-conditioned matrix inversion or solution of a descriptor control problem (no method still exists for the partial pole-placement in descriptor systems). The IMSC approach requires complete knowledge of the spectrum and the associated eigenvectors of the quadratic pencil (1) for decoupling of the open-loop pencil. Furthermore, the decoupling of the closed-loop pencil requires some very stringent conditions on actuators and sensors [1], which is unpractical for real-life applications.

In several recent papers [5–10] numerically effective methods have been developed for both the partial pole-placement and eigenstructure assignment problems; they overcome the difficulties associated with the above two approaches. These methods are designed directly in matrix second-order setting without resorting to first-order transformations and without requiring complete knowledge of the spectrum of the pencil $F(\lambda)$, as needed by the IMSC approach [1]. Although they satisfy control design requirements and are practical for control applications, unfortunately, they are not capable of preserving the symmetry of the original model.

In this paper, a novel *symmetry preserving partial spectrum assignment method* for vibrating system (1) is proposed. Specifically, the following problem is solved:

Let $\{\lambda_i\}_{i=1}^{2n}$ and $\{y_i\}_{i=1}^{2n}$ be, respectively, the spectrum and the eigenvector set of $F(\lambda)$. Given (i) symmetric $n \times n$ matrices M, C, and K of the pencil (2) with $M > 0,\ K \geqslant 0$, and $C = C^{\mathrm{T}}$, (ii) a part of the spectrum $\{\lambda_1, \ldots, \lambda_r\},\ r \leqslant 2n$ of $F(\lambda)$ and the corresponding eigenvectors $\{y_1, \ldots, y_r\}$, and (iii) a set of r complex conjugate numbers $\{\mu, \ldots, \mu_r\}$. Assuming the both sets $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_r\}$ are closed under complex conjugations, find real symmetric matrices $M_{\text{new}}, C_{\text{new}}$, and K_{new} such that the spectrum of $F_{\text{new}} = \lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}$ is $\{\mu_1, \ldots, \mu_r;\ \lambda_{r+1}, \ldots, \lambda_{2n}\}$ andfurthermore, the eigenvectors corresponding to $\lambda_{r+1}, \ldots, \lambda_{2n}$ remain unchanged. Furthermore, characterize the eigenvectors of F_{new} corresponding to $\mu_1, \ldots, \mu_r$.

The last property is highly significant from practical applications view points. It says that certain important physical properties of the system are completely preserved by updating. However, the most important benefits obtained by this new method over the existing non-symmetric pole-placement methods for the second-order model are that the *updated model remains symmetric and the changes made in the data matrices M, K, and C might be significantly less than those obtained by the pole-placement algorithms usingg feedback control.*

To distinguish this problem from the partial pole-placement problem in control theory, we will call this problem "Eigenvalue Embedding" Problem (EEP). Our major contributions to EEP in this paper are as follows:

(i) An algorithm and associated theories are developed, using low-rank symmetric updates.
(ii) Computable error bounds are derived by means of rigorous error analysis.
(iii) The accuracy of the algorithm is demonstrated by both an illustrative, and a real-life example with simulated data from the Boeing Company.
(iv) A complete characterization of the eigenvectors of the updated model is also given. It is shown by mathematical proofs that the eigenvectors corresponding to the eigenvalues which are not reassigned also remain invariant.

Finally, it is noted that the EEP addressed in this paper is clearly related to the well-known problem in vibrating engineering, called "Finite-Element Model Updating Problem" (FEMUP). The FEMUP is concerned with updating a symmetric FEM in such a way that the updated model remains symmetric and a set of measured eigenvalues and eigenvectors are incorporated into the updated model, while the other eigenvalues and eigenvectors remain invariant or at least do not spill over the regions of resonance and instability.

The problem has been well-studied: a couple of hundred papers and a book [11] have been published on the problem. For an extensive list of papers on this topic, see the reference list of the book [11]. The existing so-called "*direct methods*" [11–15] can reproduce the given set of measured data, but cannot guarantee that the remaining eigenvalues and eigenvectors of the FEM remain unchanged. Furthermore, these methods deal with undamped systems only; thus the underlying eigenvalue problem in this setting is a generalized eigenvalue problem in the liner pencil $K - \lambda M$ [2,16] rather than quadratic eigenvalue problem for the pencil (2). The quadratic eigenvalue problem is much harder to solve numerically [17].

The solution proposed in this paper for EEP can be considered as a partial but meaningful solution to the FEMUP. In contrast with the existing direct methods for FEMUP, the proposed method deals with the damped second-order model and can guarantee mathematically that the eigenvalues and eigenvectors that do not participate in the updating process remain unchanged.

2. Embedding of a real eigenvalue

In this section, we construct the updated matrices M_{new}, K_{new} and C_{new}, such that a distinct real eigenpair (λ_1, y_1) of the pencil $F(\lambda) = \lambda^2 M + \lambda C + K$ is replaced by (μ_1, y_1), where μ_1 is preassigned; $\mu_1 \neq \lambda_1$, and the other eigenvalues and eigenvectors remain invariant. To achieve this goal, we consider a low-rank transformation, called the *non-equivalence transformation* for the quadratic matrix pencil $F(\lambda)$. A non-equivalence transformation for the rational λ-matrix functions has been previously considered in Refs. [18–23]. However, the non-equivalence transformation reported in this paper cannot be derived by using a straightforward generalization of the results in the above papers.

Since (λ_1, y_1) is a real eigenpair of $F(\lambda)$, we have

$$F(\lambda_1)y_1 \equiv (\lambda_1^2 M + \lambda_1 C + K)y_1 = 0. \tag{3}$$

Since K is positive definite, the eigenvector y_1 can be normalized such that $y_1^\top K y_1 = 1$. Suppose that $\lambda_1 \in \mathbb{R}$ is a distinct unwanted eigenvalue that needs to be replaced by a prescribed real number μ_1. The following theorem provides a non-equivalence transformation of $F(\lambda)$ such that the updated matrix pencil, $F_{\text{new}}(\lambda)$, keeps the eigenstructure of $F(\lambda)$ except that μ_1 replaces λ_1 to become an eigenvalue of $F_{\text{new}}(\lambda)$.

Theorem 1 (*Real eigenvalue embedding*). *Let* (λ_1, y_1) *be a distinct real eigenpair of* $F(\lambda)$ *with* $y_1^\top K y_1 = 1$, *suppose* $\mu_1 \neq \lambda_1$, *and*

$$1 - \lambda_1 \mu_1 \theta_1 \neq 0 \quad \textit{and} \quad 1 - \lambda_1^2 \theta_1 \neq 0$$

and define

$$\theta_1 = y_1^\top M y_1, \tag{4}$$

$$\varepsilon_1 = \frac{\lambda_1 - \mu_1}{1 - \lambda_1 \mu_1 \theta_1}. \tag{5}$$

Then the updated matrix pencil

$$F_{\text{new}}(\lambda) = \lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}, \tag{6}$$

where

$$\begin{aligned} M_{\text{new}} &= M - \varepsilon_1 \lambda_1 M y_1 y_1^\top M, \\ C_{\text{new}} &= C + \varepsilon_1 (M y_1 y_1^\top K + K y_1 y_1^\top M), \\ K_{\text{new}} &= K - \frac{\varepsilon_1}{\lambda_1} K y_1 y_1^\top K \end{aligned} \tag{7}$$

is symmetric, and has the following spectral properties:

(a) *The number μ_1 is in the spectrum of $F_{\text{new}}(\lambda)$ and the remaining eigenvalues of $F_{\text{new}}(\lambda)$ are the same as those of $F(\lambda)$.*
(b) (i) *y_1 is also an eigenvector of $F_{\text{new}}(\lambda)$ corresponding to the eigenvalue μ_1.* (ii) *The remaining eigenvectors of $F_{\text{new}}(\lambda)$ are the same as those of $F(\lambda)$; that is, if $\lambda_2 \neq \lambda_1$ and (λ_2, y_2) is an eigenpair of $F(\lambda)$, then it is also an eigenpair of $F_{\text{new}}(\lambda)$.*

Proof. (a) Substituting the result of Eq. (3) into $F(\lambda)$, we obtain

$$\begin{aligned} F(\lambda) y_1 &= \lambda^2 M y_1 + \lambda C y_1 + K y_1 \\ &= \lambda^2 M y_1 + \lambda C y_1 - \lambda_1^2 M y_1 - \lambda_1 C y_1 \\ &= (\lambda - \lambda_1)((\lambda + \lambda_1) M + C) y_1. \end{aligned} \tag{8}$$

By using the identity

$$\det(I_n + RS) = \det(I_m + SR), \tag{9}$$

where $R \in \mathbb{C}^{n \times m}$ and $S \in \mathbb{C}^{m \times n}$, together with Eq. (8), we have

$$\begin{aligned} \det(F_{\text{new}}(\lambda)) &= \det(\lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}) \\ &= \det\Big(\lambda^2 M + \lambda C + K - \lambda^2 \varepsilon_1 \lambda_1 M y_1 y_1^\top M \\ &\qquad + \lambda \varepsilon_1 (M y_1 y_1^\top K + K y_1 y_1^\top M) - \frac{\varepsilon_1}{\lambda_1} K y_1 y_1^\top K\Big) \\ &= \det(F(\lambda) + \varepsilon_1 ((\lambda + \lambda_1) M + C) y_1 y_1^\top (K - \lambda \lambda_1 M)) \\ &= \det\left(F(\lambda) + \frac{\varepsilon_1}{\lambda - \lambda_1} F(\lambda) y_1 y_1^\top (K - \lambda \lambda_1 M)\right) \\ &= \det(F(\lambda))\left(1 + \frac{\varepsilon_1}{\lambda - \lambda_1} (1 - \lambda \lambda_1 \theta_1)\right) \\ &= \frac{\det(F(\lambda))}{\lambda - \lambda_1} (\lambda - \lambda_1 + \varepsilon_1 (1 - \lambda \lambda_1 \theta_1)). \end{aligned}$$

Since $1-\lambda_1^2\theta_1 \neq 0$, we now use Eq. (5) to get

$$\lambda-\lambda_1+\varepsilon_1(1-\lambda\lambda_1\theta_1)=(\lambda-\mu_1)\frac{(1-\lambda_1^2\theta_1)}{1-\lambda_1\mu_1\theta_1}.$$

Therefore, we conclude that $\det(F_{\text{new}}(\lambda))$ has the same roots as $\det(F(\lambda))$, except that λ_1 is replaced by μ_1.

(b) We first prove (b)(i). From Eq. (7), we have

$$\begin{aligned} F_{\text{new}}(\mu_1)y_1 &= \mu_1^2(M-\varepsilon_1\lambda_1 My_1y_1^\top M)y_1+\mu_1(C+\varepsilon_1 \\ &\quad \times(My_1y_1^\top K+Ky_1y_1^\top M))y_1+\left(K-\frac{\varepsilon_1}{\lambda_1}Ky_1y_1^\top K\right)y_1 \qquad (10)\\ &= (\mu_1^2-\mu_1^2\varepsilon_1\lambda_1\theta_1+\mu_1\varepsilon_1)My_1+\mu_1 Cy_1+\left(\mu_1\varepsilon_1\theta_1+1-\frac{\varepsilon_1}{\lambda_1}\right)Ky_1. \qquad (11)\end{aligned}$$

Again using Eq. (5), we have

$$\mu_1\varepsilon_1\theta_1+1-\frac{\varepsilon_1}{\lambda_1}=\varepsilon_1\left(\frac{\lambda_1\mu_1\theta_1-1}{\lambda_1}\right)+1=\frac{\mu_1}{\lambda_1}. \qquad (12)$$

Since $F(\lambda_1)y_1=0$, we have

$$Ky_1=-\lambda_1^2My_1-\lambda_1Cy_1. \qquad (13)$$

Substituting Eqs. (12) and (13) into Eq. (10), we then obtain

$$F_{\text{new}}(\mu_1)y_1=(\mu_1^2-\mu_1^2\varepsilon_1\lambda_1\theta_1+\mu_1\varepsilon_1-\lambda_1\mu_1)My_1.$$

Once more, from Eq. (5), we conclude that

$$\begin{aligned}\mu_1^2-\mu_1^2\varepsilon_1\lambda_1\theta_1+\mu_1\varepsilon_1-\lambda_1\mu_1 &= \mu_1(\mu_1-\lambda_1)+\mu_1\varepsilon_1(1-\mu_1\lambda_1\theta_1)\\ &= \mu_1(\mu_1-\lambda_1)+\mu_1(\lambda_1-\mu_1)\\ &= 0.\end{aligned}$$

This implies that $F_{\text{new}}(\mu_1)y_1=0$, and so (b)(i) is proven.

To prove (b)(ii), we observe that

$$F(\lambda_2)y_2=(\lambda_2^2M+\lambda_2C+K)y_2=0,$$

that is, $Ky_2=-\lambda_2^2My_2-\lambda_2Cy_2$. This implies

$$F(\lambda_1)y_2=(\lambda_1-\lambda_2)((\lambda_1+\lambda_2)M+C)y_2. \qquad (14)$$

Using the same arguments as in the proof of (a) and Eq. (14), we obtain

$$\begin{aligned} F_{\text{new}}(\lambda_2)y_2 &= (\lambda_2^2 M_{\text{new}} + \lambda_2 C_{\text{new}} + K_{\text{new}})y_2 \\ &= F(\lambda_2)y_2 + \frac{\varepsilon_1}{\lambda_2 - \lambda_1}(F(\lambda_2)y_1y_1^\top(K - \lambda_2\lambda_1 M)y_2) \\ &= \frac{\varepsilon_1}{\lambda_2 - \lambda_1}(F(\lambda_2)y_1y_1^\top(-\lambda_2((\lambda_1 + \lambda_2)M + C))y_2) \\ &= \frac{-\lambda_2\varepsilon_1}{(\lambda_2 - \lambda_1)^2}(F(\lambda_2)y_1y_1^\top F(\lambda_1)y_2) \\ &= 0. \end{aligned}$$

Hence, (λ_2, y_2) is also an eigenpair of $F_{\text{new}}(\lambda)$. □

Remarks. (i) Note that if $\lambda_1 = \mu_1$, then $\varepsilon_1 = 0$, and there will be no updating at all. Of course, in practice, it does not make any sense to reassign an eigenvalue which is not desirable to have in the spectrum.

(ii) An alternative and shorter proof of Theorem 1, using orthogonality relations between the eigenvectors of a symmetric positive semi-definite pencil, appear in the Ph.D. Dissertation of Carvalho [24] (available from the website: www.math.niu.edu/~dattab).

3. Embedding of a complex conjugate pair of eigenvalues

We now develop the results in this section, analogous to those of Theorem 1, to show how to compute the updated symmetric matrices M_{new}, K_{new} and C_{new}, such that a distinct complex conjugate pair of eigenvalues, μ_1 and $\bar{\mu}_1$ is assigned to the spectrum of $F_{\text{new}}(\lambda)$, while the other eigenvalues of $F_{\text{new}}(\lambda)$ and the corresponding eigenvectors remain the same as those of $F(\lambda)$. *We also give a characterization of the eigenvectors associated with the complex conjugate pair that is reassigned.* For simplicity, a matrix pair (Λ, Y) satisfying

$$MY\Lambda^2 + CY\Lambda + KY = 0$$

will be called an eigenpair of $F(\lambda)$. The notation spec (T) stands for spectrum of the matrix T.

Let (λ_1, y_1) be a complex eigenpair of $F(\lambda)$, associated with a distinct eigenvalue $\lambda_1 = \alpha_1 + \mathrm{i}\beta_1$, $\alpha_1,\ \beta_1 \in \mathbb{R},\ \beta_1 \neq 0$, and $y_1 = y_{1r} + \mathrm{i}y_{1i},\ y_{1r},\ y_{1i} \in \mathbb{R}^n$. Suppose that y_{1r} and y_{1i} are linearly independent, then y_1 and $\bar{y}_1$ are linearly independent, and $(\bar{\lambda}_1, \bar{y}_1)$ is also an eigenpair of $F(\lambda)$. Since (λ_1, y_1) is an eigenpair of $F(\lambda)$, we have

$$MZ_1\underline{\Lambda}_1^2 + CZ_1\underline{\Lambda}_1 + KZ_1 = 0, \tag{15}$$

where

$$\underline{\Lambda}_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \quad \text{and} \quad Z_1 = [y_{1r}\ \ y_{1i}].$$

Thus, $(\underline{\Lambda}_1, Z_1)$ is an eigenpair of $F(\lambda)$. Since K is positive definite, $\Sigma_1 = Z_1^\top K Z_1$ is also positive definite. Thus there exists an orthogonal matrix $S_1 \in \mathbb{R}^{2\times 2}$, and a positive diagonal matrix

$$D_1 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix},$$

such that

$$\Sigma_1 = S_1 D_1 D_1 S_1^\top.$$

Therefore, the definitions

$$Y_1 = Z_1 S_1 D_1^{-1}, \tag{16}$$

$$\Lambda_1 = D_1 S_1^\top \underline{\Lambda}_1 S_1 D_1^{-1}, \tag{17}$$

clearly imply

$$Y_1^\top K Y_1 = I_2,$$

$$\Lambda_1 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1/d \\ -d\beta_1 & \alpha_1 \end{bmatrix},$$

where $d = d_1/d_2$.

To present our main result, we need the following Lemma.

Lemma 2. *Given a complex number, $\mu_1 = \varphi_1 + \mathrm{i}\psi_1$, $\psi_1 \neq 0$, there is a real diagonal matrix, E_M, such that μ_1 is an eigenvalue of the matrix pair*

$$(\Lambda_1 \Lambda_1^\top - E_M,\ \Lambda_1^\top - E_M \Theta_1 \Lambda_1^\top),$$

where $\Theta_1 = Y_1^\top M Y_1$ and Y_1, Λ_1 are given by Eqs. (16) *and* (17), *respectively.*

Proof. Let

$$\Theta_1 = Y_1^\top M Y_1 = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{bmatrix} \quad \text{and} \quad E_M = \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \in \mathbb{R}^{2\times 2},$$

where ξ, η are two unknowns. By expanding the following two conjugated equations:

$$\begin{aligned} &\det[\mu_1(\Lambda_1^\top - E_M\Theta_1\Lambda_1^\top) - (\Lambda_1\Lambda_1^\top - E_M)] = 0, \\ &\det[\bar{\mu}_1(\Lambda_1^\top - E_M\Theta_1\Lambda_1^\top) - (\Lambda_1\Lambda_1^\top - E_M)] = 0, \end{aligned} \tag{18}$$

we conclude that ξ, η satisfy a system of two real two degree polynomials

$$\begin{aligned} &p_1 + p_2\xi + p_3\eta + p_4\xi\eta = 0, \\ &q_1 + q_2\xi + q_3\eta + q_4\xi\eta = 0, \end{aligned} \tag{19}$$

where

$$
\begin{aligned}
p_1 &= 2(\varphi_1\rho_1 - \alpha_1\sigma_1),\\
p_2 &= \sigma_1\left(\alpha_1\theta_{11} + \frac{\alpha_1}{\rho_1} + d\beta_1\theta_{12}\right) - \frac{2\varphi_1}{\rho_1}(\alpha_1^2 + d^2\beta_1^2),\\
p_3 &= \sigma_1\left(\frac{\alpha_1}{\rho_1} + \alpha_1\theta_{22} - \frac{\beta_1\theta_{12}}{d}\right) - \frac{2\varphi_1}{\rho_1}\left(\alpha_1^2 + \frac{\beta_1^2}{d^2}\right),\\
p_4 &= \frac{\sigma_1}{\rho_1}\left(d\beta_1\theta_{12} - \frac{\beta_1\theta_{12}}{d} - \alpha_1\theta_{11} - \alpha_1\theta_{22}\right) + \frac{2\varphi_1}{\rho_1},\\
q_1 &= \sigma_1 - \rho_1,\\
q_2 &= \frac{1}{\rho_1}(\alpha_1^2 + d^2\beta_1^2) - \sigma_1\theta_{11},\\
q_3 &= \frac{1}{\rho_1}\left(\alpha_1^2 + \frac{\beta_1^2}{d^2}\right) - \sigma_1\theta_{22},\\
q_4 &= \sigma_1(\theta_{11}\theta_{22} - \theta_{12}^2) - \frac{1}{\rho_1}.
\end{aligned}
$$

Here, $\theta_{j,k}$ is the (j,k)th entry of Θ_1, $j, k = 1,2$; $\rho_1 = \alpha_1^2 + \beta_1^2$ and $\sigma_1 = \varphi_1^2 + \psi_1^2$. Hence, from Eq. (19), we can find E_M by setting

$$
\xi = -\frac{q_1 + q_3\eta}{q_2 + q_4\eta} \quad \text{and} \quad \eta = \frac{-\ell_2 \pm \sqrt{\ell_2^2 - 4\ell_1\ell_3}}{2\ell_1}, \tag{20}
$$

where $\ell_1 = p_3q_4 - p_4q_3$, $\ell_2 = p_1q_4 - p_2q_3 + p_3q_2 - p_4q_1$ and $\ell_3 = p_1q_2 - p_2q_1$. $\square$

Remark 3.1. (i) It is easily seen from above that ξ and η are real provided that $\ell_2^2 - 4\ell_1\ell_3 \geqslant 0$. This will always happen whenever the assumptions of Lemma 2 hold. (ii) Formula (20) usually will give two possibly solution pairs (ξ, η). The pair (ξ, η) that gives the smaller matrix norm $\|E_M\|$ should be chosen in a numerical implementation.

The next theorem provides a low-rank transformation of the matrix pencil $F(\lambda)$, such that the eigenvalues of the updated symmetric pencil $F_{\text{new}}(\lambda)$ are the same as those of $F(\lambda)$, except for the complex pair of eigenvalues $(\lambda_1, \bar{\lambda}_1)$ of $F(\lambda)$ that is replaced by a prescribed complex pair of numbers $(\mu_1, \bar{\mu}_1)$.

Theorem 3 (*Embedding of a pair of complex conjugate eigenvalues*). *Let* Y_1 *and* Λ_1 *be the same as those defined in Eqs.* (16) *and* (17). *Let* E_M *be the same as in Lemma* 2. *Define*

$$
\begin{aligned}
M_{\text{new}} &= M - MY_1E_MY_1^\top M,\\
C_{\text{new}} &= C + MY_1E_CY_1^\top K + KY_1E_C^\top Y_1^\top M,\\
K_{\text{new}} &= K - KY_1E_KY_1^\top K,
\end{aligned} \tag{21}
$$

where

$$E_K = \Lambda_1^{-1} E_M \Lambda_1^{-\top} \quad \textit{and} \quad E_C = E_M \Lambda_1^{-\top}. \tag{22}$$

Then the real symmetric pencil $F_{\text{new}}(\lambda) = \lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}$ *has the following properties*:

(i) *The eigenvalues of the matrix pencil* $F_{\text{new}}(\lambda)$ *are the same as those of* $F(\lambda)$ *except that the pair of complex conjugate eigenvalues* $\lambda_1, \bar{\lambda}_1$ *of* $F(\lambda)$ *are replaced by the complex conjugate numbers* $\mu_1, \bar{\mu}_1$.
(ii) *The eigenvectors associated with the other eigenvalues remain the same as those of the original pencil.*
(iii) *The eigenvector associated with* μ_1 *is given by* $\bar{y}_1 = Y_1 X_1 e_1$, *where* X_1 *is a non-singular matrix that diagonalizes the matrix* $\begin{pmatrix} \phi_1 & \psi_1 \\ -\psi_1 & \phi_1 \end{pmatrix}$, *and* e_1 *is the first unit vector.* (*Note that* $\mu_1 = \phi_1 + \mathrm{i}\psi_1$.)

Proof. (i) From Eq. (15) and the definitions of Y_1 and Λ_1, we see that (Λ_1, Y_1) is an eigenpair of $F(\lambda)$ and therefore

$$MY_1\Lambda_1^2 + CY_1\Lambda_1 + KY_1 = 0.$$

Now, letting $\Lambda = \lambda I_2$, we have

$$\begin{aligned} F(\lambda)Y_1 &= (\lambda^2 M + \lambda C + K)Y_1 \\ &= (MY_1(\Lambda + \Lambda_1) + CY_1)(\Lambda - \Lambda_1). \end{aligned} \tag{23}$$

From Eqs. (21)–(23), we obtain

$$\begin{aligned} F_{\text{new}}(\lambda) &= \lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}} \\ &= F(\lambda) + \lambda M Y_1 E_C Y_1^\top K - K Y_1 E_K Y_1^\top K + \lambda K Y_1 E_C^\top Y_1^\top M \\ &\quad - \lambda^2 M Y_1 E_M Y_1^\top M \\ &= F(\lambda) + (CY_1 + MY_1(\Lambda + \Lambda_1))\Lambda_1 E_K (Y_1^\top K - \lambda \Lambda_1^\top Y_1^\top M) \\ &= F(\lambda) + F(\lambda) Y_1 (\Lambda - \Lambda_1)^{-1} \Lambda_1 E_K (Y_1^\top K - \lambda \Lambda_1^\top Y_1^\top M). \end{aligned}$$

This implies

$$\begin{aligned} \det[F_{\text{new}}(\lambda)] &= \det[F(\lambda) + F(\lambda) Y_1 (\Lambda - \Lambda_1)^{-1} \Lambda_1 E_K (Y_1^\top K - \lambda \Lambda_1^\top Y_1^\top M)] \\ &= \det[F(\lambda)] \det[I_n + Y_1 (\Lambda - \Lambda_1)^{-1} \Lambda_1 E_K (Y_1^\top K - \lambda \Lambda_1^\top Y_1^\top M)] \\ &= \det[F(\lambda)] \det[I_2 + (\Lambda - \Lambda_1)^{-1} \Lambda_1 E_K (I_2 - \lambda \Lambda_1^\top \Theta_1)] \\ &= \frac{\det[F(\lambda)]}{(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)} \det[(\lambda I_2 - \Lambda_1) + \Lambda_1 E_K (I_2 - \lambda \Lambda_1^\top \Theta_1)] \\ &= \frac{\det[F(\lambda)]}{(\lambda - \lambda_1)(\lambda - \bar{\lambda}_1)} \det[\lambda (I_2 - \Lambda_1 E_K \Lambda_1^\top \Theta_1) - \Lambda_1 (I_2 - E_K)]. \end{aligned}$$

By Lemma 3.1, the pair $\{\mu_1, \bar{\mu}_1\}$ is a complex conjugate pair of the eigenvalues of the matrix pencil $(\Lambda_1\Lambda_1^\top - E_M, \Lambda_1^\top - E_M\Theta_1\Lambda_1^\top)$. This implies that

$$\det[\lambda(I_2 - \Lambda_1 E_K \Lambda_1^\top \Theta_1) - \Lambda_1(I_2 - E_K)] = \frac{(\lambda - \mu_1)(\lambda - \bar{\mu}_1)}{\lambda_1\bar{\lambda}_1}.$$

Therefore, $F_{\text{new}}(\lambda)$ has the same eigenvalues as $F(\lambda)$, except that λ_1, $\bar{\lambda}_1$ are now replaced by μ_1, $\bar{\mu}_1$.

(ii) Let $\lambda_2 = \alpha_2 + \mathrm{i}\beta_2$ and $y_2 = y_{2r} + \mathrm{i}y_{2i}$. Define Y_2 and Λ_2 in the same way as Y_1 and Λ_1 have been defined in Eqs. (15)–(17). Then (Λ_1, Y_1) and (Λ_2, Y_2) are eigenpairs of $F(\lambda)$, with $Y_1^\top KY_1 = I_2$ and $Y_2^\top KY_2 = I_2$. Thus

$$Y_2^\top KY_1 + Y_2^\top CY_1\Lambda_1 + Y_2^\top MY_1\Lambda_1^2 = 0, \tag{24}$$

$$Y_2^\top KY_1 + \Lambda_2^\top Y_2^\top CY_1 + (\Lambda_1^\top)^2 Y_2^\top MY_1 = 0. \tag{25}$$

Eliminating the terms involving "$Y_2^\top CY_1$" from Eqs. (24) and (25), we have

$$\Lambda_2^\top(Y_2^\top KY_1) - (Y_2^\top KY_1)\Lambda_1 + \Lambda_2^\top(Y_2^\top MY_1)\Lambda_1^2 - (\Lambda_2^\top)^2(Y_2^\top MY_1)\Lambda_1 = 0.$$

Let $K_Y = Y_2^\top KY_1$, $M_Y = Y_2^\top MY_1$. Let $\otimes$ and $\mathrm{vec}(\cdot)$ denote the Kronecker product and vectorizing operator, respectively. Then vectorizing the last equation, we have

$$\begin{aligned}(I \otimes \Lambda_2^\top - \Lambda_1^\top \otimes I)\mathrm{vec}(K_Y) &= \mathrm{vec}(\Lambda_2^\top(\Lambda_2^\top M_Y - M_Y\Lambda_1)\Lambda_1) \\ &= (\Lambda_1^\top \otimes \Lambda_2^\top)\mathrm{vec}(\Lambda_2^\top M_Y - M_Y\Lambda_1) \\ &= (\Lambda_1^\top \otimes \Lambda_2^\top)(I \otimes \Lambda_2^\top - \Lambda_1^\top \otimes I)\mathrm{vec}(M_Y) \\ &= (I \otimes \Lambda_2^\top - \Lambda_1^\top \otimes I)(\Lambda_1^\top \otimes \Lambda_2^\top)\mathrm{vec}(M_Y).\end{aligned}$$

Suppose $\lambda_1 \neq \lambda_2$, then $\mathrm{spec}(\Lambda_1) \cap \mathrm{spec}(\Lambda_2) = \emptyset$. This implies that the matrix $(I \otimes \Lambda_2^\top - \Lambda_1^\top \otimes I)$ is non-singular and hence, $(\Lambda_1^\top \otimes \Lambda_2^\top)(\mathrm{vec}(M_Y)) = \mathrm{vec}(K_Y)$. Thus,

$$Y_2^\top KY_1 = \Lambda_2^\top(Y_2^\top MY_1)\Lambda_1. \tag{26}$$

Since (Λ_2, Y_2) is an eigenpair of $F(\lambda)$, we have

$$MY_2\Lambda_2^2 + CY_2\Lambda_2 + KY_2 = 0.$$

From Eq. (21) it then follows that

$$\begin{aligned}&M_{\text{new}}Y_2\Lambda_2^2 + C_{\text{new}}Y_2\Lambda_2 + K_{\text{new}}Y_2 \\ &\quad = (M - MY_1E_MY_1^\top M)Y_2\Lambda_2^2 + (C + MY_1E_CY_1^\top K + KY_1E_C^\top Y_1^\top M)Y_2\Lambda_2 \\ &\qquad + (K - KY_1E_KY_1^\top K)Y_2 \\ &\quad = MY_2\Lambda_2^2 - MY_1E_MY_1^\top MY_2\Lambda_2^2 + CY_2\Lambda_2 + MY_1E_CY_1^TKY_2\Lambda_2 \\ &\qquad + KY_1E_C^\top Y_1^\top MY_2\Lambda_2 + KY_2 - KY_1E_KY_1^TKY_2 \\ &\quad = -MY_1\Lambda_1E_K\Lambda_1^\top Y_1^\top MY_2\Lambda_2^2 + MY_1\Lambda_1E_KY_1^\top KY_2\Lambda_2 + KY_1E_k\Lambda_1^\top Y_1^\top \\ &\qquad \times MY_2\Lambda_2 - KY_1E_KY_1^\top KY_2 \\ &\quad = MY_1\Lambda_1E_K(Y_1^\top KY_2\Lambda_2 - \Lambda_1^\top Y_1^\top MY_2\Lambda_2^2) + KY_1E_K(\Lambda_1^\top Y_1^\top MY_2\Lambda_2 - Y_1^\top KY_2).\end{aligned}$$

Using Eq. (26), we then obtain that $M_{\text{new}}Y_2\Lambda_2^2 + C_{\text{new}}Y_2\Lambda_2 + K_{\text{new}}Y_2 = 0$.

(iii) Let $\underline{\Omega}_1 = \begin{bmatrix} \varphi_1 & \psi_1 \\ -\psi_1 & \varphi_1 \end{bmatrix}$, where $\mu_1 = \varphi_1 + i\psi_1$ is a complex eigenvalue of $F_{\text{new}}(\lambda)$, with $\psi_1 \neq 0$. From Eq. (18), there exists a non-singular matrix $V_1 \in \mathbb{R}^{2\times 2}$ such that

$$(I_2 - \Lambda_1 E_K \Lambda_1^\top \Theta_1) V_1 \underline{\Omega}_1 - \Lambda_1 (I_2 - E_K) V_1 = 0.$$

By setting $\Omega_1 = V_1 \underline{\Omega}_1 V_1^{-1}$, we obtain

$$(I_2 - \Lambda_1 E_K \Lambda_1^\top \Theta_1) \Omega_1 - \Lambda_1 (I_2 - E_K) = 0. \tag{27}$$

Now,

$$\begin{aligned} M_{\text{new}} Y_1 \Omega_1^2 + C_{\text{new}} Y_1 \Omega_1 + K_{\text{new}} Y_1 = {} & M Y_1 (\Omega_1^2 - E_M \Theta_1 \Omega_1^2 + E_C \Omega_1) + C Y_1 \Omega_1 \\ & + K Y_1 (E_C^\top \Theta_1 \Omega_1 + I_2 - E_K), \end{aligned} \tag{28}$$

where $\Theta_1 = Y_1^\top M Y_1$. Since (Λ_1, Y_1) is an eigenpair of $F(\lambda)$, and Ω_1 satisfies Eq. (27), we conclude that

$$\begin{aligned} & C Y_1 \Omega_1 + K Y_1 (E_C^\top \Theta_1 \Omega_1 + I_2 - E_K) \\ & = C Y_1 \Omega_1 + (-M Y_1 \Lambda_1^2 - C Y_1 \Lambda_1)(E_K \Lambda_1^\top \Theta_1 \Omega_1 + I_2 - E_K) \\ & = -M Y_1 \Lambda_1^2 (E_K \Lambda_1^\top \Theta_1 \Omega_1 + I_2 - E_K) + C Y_1 [(I_2 - \Lambda_1 E_K \Lambda_1^\top \Theta_1)\Omega_1 - \Lambda_1 (I_2 - E_K)] \\ & = -M Y_1 \Lambda_1^2 (E_K \Lambda_1^\top \Theta_1 \Omega_1 + I_2 - E_K). \end{aligned}$$

Therefore, Eq. (28) becomes

$$\begin{aligned} & M Y_1 [\Omega_1^2 - E_M \Theta_1 \Omega_1^2 + E_C \Omega_1 - \Lambda_1^2 (E_K \Lambda_1^\top \Theta_1 \Omega_1) - \Lambda_1^2 (I_2 - E_K)] \\ & \quad = M Y_1 [((I_2 - E_M \Theta_1)\Omega_1 - \Lambda_1 (I_2 - E_K))\Omega_1 + \Lambda_1 ((I_2 - E_M \Theta_1) \\ & \quad\quad \times \Omega_1 - \Lambda_1 (I_2 - E_K))] \\ & \quad = 0. \end{aligned}$$

Thus, (Y_1, Ω_1) is an eigenpair of $F_{\text{new}}(\lambda)$. Letting $T_1 \Omega_1 T_1^{-1} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \bar{\mu}_1 \end{pmatrix}$ and setting $X = T_1 V_1$, the (iii) is proved. □

Based on the above theorem, we present the following algorithm for assigning a pair of complex conjugate numbers to be eigenvalues of the updated symmetric matrix pencil.

Algorithm 3.1 (*Assignment of a Pair of Complex Conjugate Eigenvalues*).
Input:

(i) An unwanted distinct complex eigenvalue, $\lambda_1 = \alpha_1 + i\beta_1$, $\alpha_1, \beta_1 \in \mathbb{R}$, $\beta_1 \neq 0$ (and its complex conjugate), and the corresponding eigenvector, $y_1 = y_{1r} + i y_{1i}$, $y_{1r}, y_{1i} \in \mathbb{R}^n$, with y_{1r}, y_{1i} being linearly independent.
(ii) A pair of complex conjugate numbers, μ_1 and $\bar{\mu}_1$, that needs to be embedded.
(iii) Symmetric matrices, M, C and K, with M, and K positive definite.

Output: Symmetric matrices M_{new}, C_{new} and K_{new} such that the updated pencil $F_{\text{new}}(\lambda) = \lambda^2 M_{\text{new}} + C_{\text{new}}\lambda + K_{\text{new}}$ has the eigenpair $\{\mu_1, \bar{\mu}_1\}$ in its spectrum, the remaining eigenvalues and eigenvectors are the same, and the eigenvector associated with μ_1 is given by $Y_1 X_1 e_1$, where Y_1 and X_1 are as defined in Theorem 3.

Step 1: Use Eqs. (16) and (17) to find the eigenpair (Λ_1, Y_1) of the original matrix pencil

$$F(\lambda) = \lambda^2 M + \lambda C + K \text{ such that } \mathrm{spec}(\Lambda_1) = \{\lambda_1, \bar{\lambda}_1\} \text{ and } Y_1^\top K Y_1 = I_2.$$

Step 2: Determine ξ and η by using formula (20).
If ξ or η is complex then stop and **return** to Step 1.
Step 3: Set $E_M = \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix}$, $E_K = \Lambda_1^{-1} E_M \Lambda_1^{-\top}$ and $E_C = E_M \Lambda_1^{-\top}$.
Step 4: Computed the updated matrices

$$M_{\text{new}} = M - MY_1 E_M Y_1^\top M,$$
$$C_{\text{new}} = C + MY_1 E_C Y_1^\top K + KY_1 E_C^\top Y_1^\top M,$$
$$K_{\text{new}} = K - KY_1 E_K Y_1^\top K.$$

Remark 3.2. Above, we have discussed how to replace an unwanted complex conjugate pair $\{\lambda_1, \bar{\lambda}_1\}$ by a prescribed conjugate pair $(\mu_1, \bar{\mu}_1)$, assuming that the associated eigenvector $y_1 = y_{1r} + \mathrm{i} y_{1i}$ is such that y_{1r} and y_{1i} are linearly independent.

We now consider the degenerate case where the real and the imaginary parts of the eigenvector, y_{1r} and y_{1i} are linearly dependent. In this case, the eigenvectors corresponding to λ_1 and $\bar{\lambda}_1$, are also linearly dependent. Hence, the eigenvector y_1 can be a real vector, i.e., $y_1 \in \mathbb{R}^n$. Since both (λ_1, y_1) and $(\bar{\lambda}_1, y_1)$ are eigenpairs of $F(\lambda)$, we have

$$\lambda_1^2 M y_1 + \lambda_1 C y_1 + K y_1 = 0,$$
$$\bar{\lambda}_1^2 M y_1 + \bar{\lambda}_1 C y_1 + K y_1 = 0.$$

Then, we obtain $(\lambda_1 + \bar{\lambda}_1) M y_1 + C y_1 = 0$. This implies that $Cy_1 // My_1$, and thus, $Ky_1 // My_1$. Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix such that $Q^\top y_1 = e_1$. Let

$$\widetilde{M} = Q^\top M Q, \quad \widetilde{C} = Q^\top C Q, \quad \widetilde{K} = Q^\top K Q,$$

then the first columns of $\widetilde{M}$, $\widetilde{C}$, $\widetilde{K}$ are mutually parallel. Furthermore, since $\widetilde{M}$, $\widetilde{C}$, $\widetilde{K}$ are symmetric, the first row vectors of $\widetilde{M}$, $\widetilde{C}$, $\widetilde{K}$ are also mutually parallel. Hence, if we apply an elementary matrix L to eliminate the second through the nth elements of the column of $\widetilde{M}$ (see Ref. [16]) then the first columns and rows of the matrices $L\widetilde{M}L^\top$, $L\widetilde{C}L^\top$, $L\widetilde{K}L^\top$ are parallel to e_1. Hence, the dimension of the quadratic problem in this case can be reduced to $n-1$ by removing the first row and column of matrices $L\widetilde{M}L^\top$, $L\widetilde{C}L^\top$, $L\widetilde{K}L^\top$simultaneously. Thus, the unwanted eigenvalues λ_1 and $\bar{\lambda}_1$ are deflated simultaneously, reducing the dimension of the problem by 1. Algorithm 3.1 now can be applied to the reduced problem.

Remark 3.3. In case $\lambda_1 = \alpha_1 + \mathrm{i}\beta_1 \in \mathbb{C}$ is a multiple eigenvalue, the formula (21) can still be used to update the matrices M, C and K; however, in this case to construct Y_1 we must consider not only the eigenvector y_1, but the associated generalized eigenvector as well. Thus, if λ_1 is an eigenvalue with multiplicity 2, then Y_1 is computed by normalizing $[y_{1r}, y_{1i}, z_{1r}, z_{1i}]$ with $Y_1^\top K Y_1 = I_4$, and y_{1r}, $z_{1r}(y_{1i}, z_{1i})$ are, respectively, the real (and imaginary) parts of y_1, z_1, and E_M is diagonal and is yet to be determined, $E_C = E_M \widetilde{\Lambda}_1^{-\top}$, $E_K = \widetilde{\Lambda}_1^{-1} E_M \widetilde{\Lambda}_1^{-\top}$. In addition, $\widetilde{\Lambda}_1 \in \mathbb{R}^{4\times 4}$ is similar to $\begin{bmatrix} \Lambda_1 & I \\ 0 & \Lambda_1 \end{bmatrix}$, and $\Lambda_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}$. The matrix E_M can be found by the following system of equations:

$$\det(F_{\text{new}}(\mu_1)) = 0, \quad \det(F_{\text{new}}(\bar{\mu}_1)) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[\det(F_{\text{new}}(\mu_1))] = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}[\det(F_{\text{new}}(\bar{\mu}_1))] = 0.$$

An Illustrative Example. Consider application of Algorithm 3.1 to a free beam with

I = Moment of inertia = $1.136 \times 10^{-9}\,\mathrm{m}^4$,
E = Young's modulus = 72 Gpa,
l = Length of the beam = 0.4005 m.

The stiffness matrix has the form

$$K = \frac{EI}{l^3}\begin{pmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{pmatrix}.$$

With the above values of I, E, and l, we have

$$K = 10^4\begin{pmatrix} 1.5330 & 0.3066 & -1.5330 & 0.3066 \\ 0.3066 & 0.0818 & -0.3066 & 0.0409 \\ -1.5330 & 0.3066 & 1.5330 & -0.3066 \\ 0.3066 & 0.0409 & -0.3066 & 0.0818 \end{pmatrix},$$

$$M = \begin{pmatrix} 0.1349 & 0.0076 & 0.0467 & -0.0045 \\ 0.0076 & 0.0006 & 0.0045 & -0.0004 \\ 0.0467 & 0.0045 & 0.1349 & -0.0076 \\ -0.0045 & -0.0004 & -0.0076 & 0.0006 \end{pmatrix},$$

$$D = 0.$$

The eigenvalues of $F(\lambda)$ are: $10^3(\pm 5.4363\mathrm{i}, \pm 1.5916\mathrm{i}, 0, 0, 0, 0)$. The pair of the complex eigenvalues, $10^3(\pm 1.5916\mathrm{i})$ were charged to $10^3(\pm 1.3509\mathrm{i})$, obtained from an experiment at the vibration laboratory at Northern Illinois University.

The updated stiffness matrix is given by

$$K_{\text{new}} = 10^4\begin{pmatrix} 1.5330 & 0.3066 & -1.5330 & 0.3066 \\ 0.3066 & 0.0787 & -0.3066 & 0.0440 \\ -1.5330 & -0.3066 & 1.5330 & -0.3066 \\ 0.3066 & 0.0440 & -0.3066 & 0.0787 \end{pmatrix}.$$

The entries of the updated mass matrix M_{new} are almost the same as those of the original matrix and the entries of the matrix D_{new} are of $O(10^{-14})$. The results on both 1 and 10 elements of the beam are displayed in the accompanying figures (Figs. 1–4).

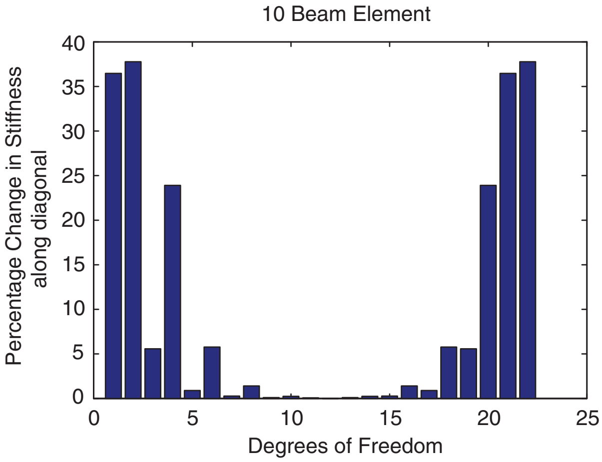

Fig. 1. Percent change in the diagonal entries of the stiffness matrix for ten beam element.

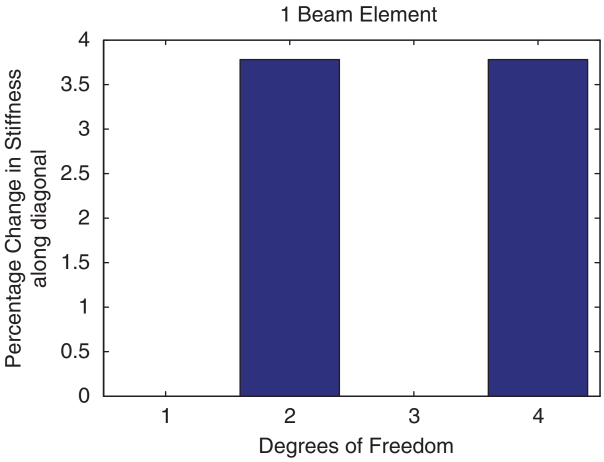

Fig. 2. Percent change in the diagonal entries of the stiffness matrix for one beam element.

4. Error analysis for the assignment of a complex conjugate pair of eigenvalues

In this section $\| \cdot \|$ denotes the 2-norm, $\hat{}$ (hat) denotes a computed quantity and the term HOT stands for "the higher-order terms."

First, we estimate the error bounds for the computed M_{new}. From Eq. (21), we have

$$\begin{aligned} \|\widehat{M}_{\text{new}} - M_{\text{new}}\| &= \|M \widehat{Y}_1 \widehat{E}_M \widehat{Y}_1^\top M - M Y_1 E_M Y_1^\top M\| \\ &\leqslant \|M\|^2 \|\widehat{Y}_1 \widehat{E}_M \widehat{Y}_1^\top - Y_1 E_M Y_1^\top\|. \end{aligned} \tag{29}$$

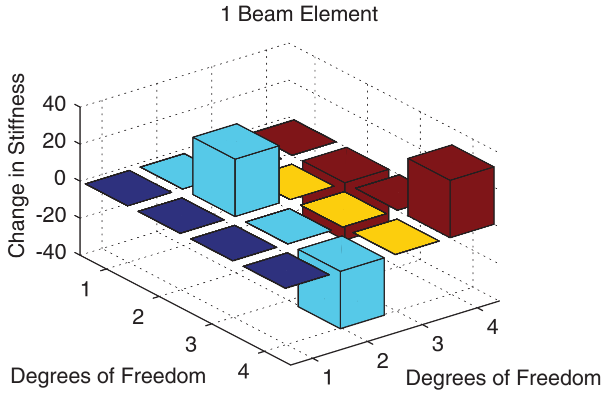

Fig. 3. Change in the entries of the stiffness matrix for one beam element.

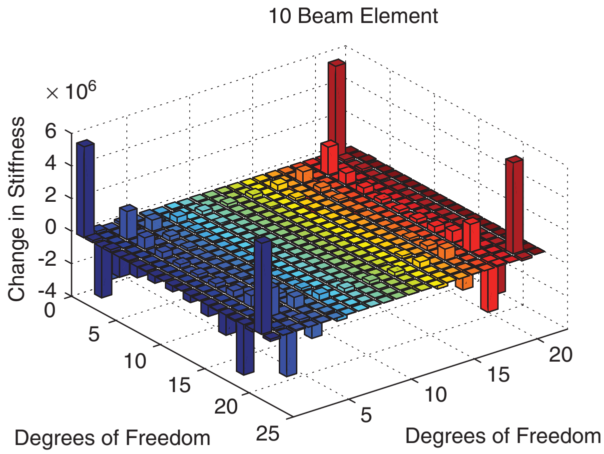

Fig. 4. Change in the entries of the stiffness matrix for 10 beam element.

By using the triangular inequality, we obtain

$$\begin{aligned}
&\|\widehat{Y}_1\widehat{E}_M\widehat{Y}_1^\top - Y_1E_MY_1^\top\|\\
&\quad\leqslant\|\widehat{Y}_1\widehat{E}_M\widehat{Y}_1^\top - \widehat{Y}_1\widehat{E}_MY_1^\top\| + \|\widehat{Y}_1\widehat{E}_MY_1^\top - \widehat{Y}_1E_MY_1^\top\|\\
&\qquad+\|\widehat{Y}_1E_MY_1^\top - Y_1E_MY_1^\top\| + \|\widehat{Y}_1 - Y_1\|\,\|E_MY_1^\top\|\\
&\quad\leqslant\|\widehat{Y}_1\widehat{E}_M\|\,\|\widehat{Y}_1 - \widehat{Y}_1 - Y_1\| + \|\widehat{Y}_1\|\,\|\widehat{Y}\|\,\|\widehat{E}_M - E_M\| + \|\widehat{Y}_1 - Y_1\|\,\|E_MY_1^\top\|\\
&\quad\leqslant[(\|\widehat{Y}_1 - Y_1\| + \|Y_1\|)\|\widehat{E}_M - E_M\| + \|\widehat{Y}_1 - Y_1\|\,\|E_M\| + \|Y_1E_M\|]\,\|\widehat{Y}_1 - Y_1\|\\
&\qquad+(\|\widehat{Y}_1 - Y_1\| + \|Y_1\|)\|Y_1\|\,\|\widehat{E}_M - E_M\| + \|\widehat{Y}_1 - Y_1\|\,\|E_MY_1^\top\|.
\end{aligned}\tag{30}$$

From the definition of Y_1 in Eq. (16), we then have

$$\|\widehat{Y}_1 - Y_1\| = \|\widehat{Z}_1\widehat{S}_1\widehat{D}_1^{-1} - Z_1S_1D_1^{-1}\| \tag{31}$$

$$\begin{aligned}&\leqslant [(\|\widehat{Z}_1 - Z_1\| + \|Z_1\|)\|\widehat{S}_1 - S_1\| + \|\widehat{Z}_1 - Z_1\|\,\|S_1\| + \|Z_1S_1\|]\|\widehat{D}_1^{-1} - D_1^{-1}\| \\ &\quad + (\|\widehat{Z}_1 - Z_1\| + \|Z_1\|)\|D_1^{-1}\|\,\|\widehat{S}_1 - S_1\| + \|\widehat{Z}_1 - Z_1\|\,\|S_1D_1^{-1}\|.\end{aligned} \tag{32}$$

It is known (see Ref. [17]) that the error bound for Z_1 satisfies

$$\|\widehat{Z}_1 - Z_1\| \leqslant c_1\varepsilon, \tag{33}$$

where

$$c_1 = \sum_{k=2}^{2n} \frac{\|z_k\|}{|\lambda_k - \lambda_1|(1 + |\overline{\lambda}_k\lambda_1|)|z_k^{\mathrm{H}}y_1|},$$

z_k and y_k are, respectively, the left and right eigenvectors corresponding to the eigenvalue λ_k of $F(\lambda)$, $Z_k = [y_{kr}\ y_{ki}]$. Similarly,

$$\|\widehat{S}_1 - S_1\| \leqslant c_2\varepsilon, \tag{34}$$

where c_2 is a constant. Since $D_1 = \left[\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right] \in \mathbb{R}^{2\times2}$, $d_1 > d_2$, and $S_1 \in \mathbb{R}^{2\times2}$ is orthogonal, we have

$$\|D_1^{-1}\| = \|S_1D_1^{-1}\| = \frac{1}{d_2} \tag{35}$$

and

$$\begin{aligned}\|\widehat{D}_1^{-1} - D_1^{-1}\| &= \|D_1^{-1}(D_1 - \widehat{D}_1)\widehat{D}_1^{-1}\| \\ &\leqslant \|D_1^{-1}\|^2\|D_1 - \widehat{D}_1\| + \mathrm{HOT} \\ &= \frac{d_1}{d_2^2}\varepsilon + \mathrm{HOT}.\end{aligned} \tag{36}$$

From Eqs. (33)–(36) and (31), we then obtain

$$\|\widehat{Y}_1 - Y_1\| \leqslant c_3\varepsilon + \mathrm{HOT}, \tag{37}$$

where

$$c_3 = \left(\frac{d_1}{d_2^2} + \frac{c_2}{d_2}\right)\|Z_1\| + \frac{c_1}{d_2}.$$

Since $E_M = \left[\begin{smallmatrix} \xi & 0 \\ 0 & \eta \end{smallmatrix}\right]$, we have

$$\|\widehat{E}_M - E_M\| = \max\{|\widehat{\xi} - \xi|,\ |\widehat{\eta} - \eta|\}. \tag{38}$$

We now obtain the bounds for $|\hat{\zeta} - \zeta|$, and $|\hat{\eta} - \eta|$.

From Eq. (20) and relations (17)–(19), we know that

$$\begin{aligned}\xi &= \xi(\lambda_1, \mu_1) = \xi(\alpha_1, \beta_1, \varphi_1, \psi_1), \\ \eta &= \eta(\lambda_1, \mu_1) = \eta(\alpha_1, \beta_1, \varphi_1, \psi_1),\end{aligned}$$

where $\lambda_1 = \alpha_1 + i\beta_1$ and $\mu_1 = \varphi_1 + i\psi_1$. In addition, we have

$$\hat{\xi} = \hat{\xi}(\alpha_1, \beta_1, \varphi_1, \psi_1) = \xi(\hat{\alpha}_1, \hat{\beta}_1, \hat{\varphi}_1, \hat{\psi}_1) + \text{HOT},$$
$$\hat{\eta} = \hat{\eta}(\alpha_1, \beta_1, \varphi_1, \psi_1) = \eta(\hat{\alpha}_1, \hat{\beta}_1, \hat{\varphi}_1, \hat{\psi}_1) + \text{HOT}.$$

So,

$$\hat{\xi} - \xi = \frac{\partial \xi}{\partial \alpha_1}\,\Delta\alpha + \frac{\partial \xi}{\partial \beta_1}\,\Delta\beta + \frac{\partial \xi}{\partial \varphi_1}\,\Delta\varphi + \frac{\partial \xi}{\partial \psi_1}\,\Delta\psi + \text{HOT}, \tag{39}$$

$$\hat{\eta} - \eta = \frac{\partial \eta}{\partial \alpha_1}\,\Delta\alpha + \frac{\partial \eta}{\partial \beta_1}\,\Delta\beta + \frac{\partial \eta}{\partial \varphi_1}\,\Delta\varphi + \frac{\partial \eta}{\partial \psi_1}\,\Delta\psi + \text{HOT}, \tag{40}$$

where $\Delta\alpha = \hat{\alpha}_1 - \alpha_1$, $\Delta\beta = \hat{\beta}_1 - \beta_1$, $\Delta\varphi = \hat{\varphi}_1 - \varphi_1$, and $\Delta\psi = \hat{\psi}_1 - \psi_1$. Since $\mu_1 = \varphi + i\psi$ is a prescribed number, we need not calculate it. The numbers $\Delta\varphi$ and $\Delta\psi$ are usually much smaller than $\Delta\alpha$ or $\Delta\beta$. We can hence ignore terms involving $\Delta\varphi$ or $\Delta\psi$ in Eqs. (39) and (40). Hence, we are only concerned with those terms related to $\Delta\alpha$ or $\Delta\beta$ in the estimation of the error bounds for ξ and η. From Eqs. (39) and (40), we have

$$\begin{aligned}|\hat{\xi} - \xi| &\leqslant \left|\frac{\partial \xi}{\partial \alpha_1}\right| |\Delta\alpha| + \left|\frac{\partial \xi}{\partial \beta_1}\right| |\Delta\beta| + \text{HOT} \\ &\leqslant \left(\left|\frac{\partial \xi}{\partial \alpha_1}\right| + \left|\frac{\partial \xi}{\partial \beta_1}\right|\right)|\hat{\lambda}_1 - \lambda_1| + \text{HOT},\end{aligned}$$

$$\begin{aligned}|\hat{\eta} - \eta| &\leqslant \left|\frac{\partial \eta}{\partial \alpha_1}\right| |\Delta\alpha| + \left|\frac{\partial \eta}{\partial \beta_1}\right| |\Delta\beta| + \text{HOT} \\ &\leqslant \left(\left|\frac{\partial \eta}{\partial \alpha_1}\right| + \left|\frac{\partial \eta}{\partial \beta_1}\right|\right)|\hat{\lambda}_1 - \lambda_1| + \text{HOT}.\end{aligned}$$

After performing some tedious calculations, it can be shown that $|\partial\xi/\partial\alpha_1|$, $|\partial\xi/\partial\beta_1|$, $|\partial\eta/\partial\alpha_1|$ and $|\partial\eta/\partial\beta_1|$ are bounded by the relative rational functions in α_1, β_1 and $|\lambda_1|$. More precisely, one can prove that

$$|\hat{\xi} - \xi| \leqslant \frac{|\zeta_1(\alpha_1, \beta_1)|}{\zeta_2(|\lambda_1|)}|\hat{\lambda}_1 - \lambda_1| + \text{HOT}, \tag{41}$$

$$|\hat{\eta} - \eta| \leqslant \frac{|\varsigma_1(\alpha_1, \beta_1)|}{\varsigma_2(|\lambda_1|)}|\hat{\lambda}_1 - \lambda_1| + \text{HOT}, \tag{42}$$

where ζ_1, ς_1 are low degree polynomials in α_1, β_1, and ζ_2, ς_2 are low-degree polynomials in $|\lambda_1|$. Since $|\lambda_1| \neq 0$, ζ_2 and ς_2 are non-zero, and both bounds in Eqs. (41) and (42) are finite. Again,

$$|\hat{\lambda}_1 - \lambda_1| \leqslant \frac{1}{(1 + |\lambda_1|^2)|z_1^{\mathrm{H}} y_1|}\varepsilon. \tag{43}$$

Substituting Eqs. (41)–(43) into Eq. (38), we have

$$\|\hat{E}_M - E_M\| = \max\{|\hat{\xi} - \xi|,\ |\hat{\eta} - \eta|\} \leqslant c_4\varepsilon + \text{HOT}, \tag{44}$$

where

$$
\begin{aligned}
c_4 &= \frac{\zeta(\alpha_1,\beta_1)}{\varsigma(|\lambda_1|)}\frac{1}{(1+|\lambda_1|^2)|z_1^{\mathrm{H}}y_1|},\\
\zeta(\alpha_1,\beta_1) &= \max\{|\zeta_1(\alpha_1,\beta_1)|,\ |\varsigma_1(\alpha_1,\beta_1)|\},\\
\varsigma(|\lambda_1|) &= \min\{\zeta_1(|\lambda_1|),\ \varsigma_1(|\lambda_1|)\}.
\end{aligned}
$$

By using Eqs. (30), (37) and (44), we then obtain

$$
\|\widehat{Y}_1\widehat{E}_M\widehat{Y}_1^{\top} - Y_1E_MY_1^{\top}\| \leqslant [2\|Y_1\|\,\|E_M\|c_3 + \|Y_1\|^2c_4]\cdot\varepsilon + \mathrm{HOT}. \tag{45}
$$

Using Eq. (45) in Eq. (29), we finally obtain the following error bound for M_{new}:

$$
\|\widehat{M}_{\text{new}} - M_{\text{new}}\| \leqslant \varepsilon\|M\|^2[2\|Y_1\|\,\|E_M\|c_3 + \|Y_1\|^2c_4]. \tag{46}
$$

To estimate the error bounds for C_{new} and K_{new}, we first need to find the error bound for Λ_1^{-1}. From Eq. (17), we have

$$
\|\widehat{\Lambda}_1^{-1} - \Lambda_1^{-1}\| = \|\widehat{D}_1\widehat{\underline{\Lambda}}_1^{-1}\widehat{D}_1^{-1} - D_1\underline{\Lambda}_1^{-1}D_1^{-1}\| \leqslant c_5\varepsilon + \mathrm{HOT},
$$

where

$$
c_5 = \frac{d_1}{d_2}\left(\frac{d_1}{d_2|\lambda_1|} + \frac{1}{|\lambda_1|} + \frac{1}{(1+|\lambda_1|^2)|z_1^{\mathrm{H}}y_1|}\right).
$$

Hence, by a similar process as above, we obtain the error bounds for C_{new} and K_{new} as given below

$$
\|\widehat{C}_{\text{new}} - C_{\text{new}}\| \leqslant 2\varepsilon\|M\|\,\|K\|\left[\|E_M\|\left(\frac{d_1c_3}{d_2|\lambda_1|} + c_5\right) + \frac{c_4}{|\lambda_1|}\right], \tag{47}
$$

$$
\|\widehat{K}_{\text{new}} - K_{\text{new}}\| \leqslant \varepsilon\|K\|^2\left[2\|Y_1\|\,\|E_M\|\frac{d_1^2c_3}{d_2^2|\lambda_1|^2} + \|Y_1\|^2\left(\frac{2d_1}{d_2|\lambda_1|}\|E_M\|c_5 + \frac{d_1^2c_4}{d_2^2|\lambda_1|^2}\right)\right]. \tag{48}
$$

5. Simultaneous assignment of several real eigenvalues

So far, we have considered the problem of assigning either one real or a pair of complex conjugate eigenvalues. In this section, we consider the simultaneous assignment of several real eigenvalues.

It is always possible to embed the sequence of real eigenvalues, $\{\mu_1,\ldots,\mu_{m_r}\}$ in the updated symmetric matrix pencil, $F_{\text{new}}(\lambda)$, by using the formula (7) recursively, for $s = 1,\ldots,m_r$

$$
\begin{aligned}
M_s &= M_{s-1} - \varepsilon_s\lambda_sM_{s-1}y_sy_s^{\top}M_{s-1},\\
C_s &= C_{s-1} + \varepsilon_s(M_{s-1}y_sy_s^{\top}K_{s-1} + K_{s-1}y_sy_s^{\top}M_{s-1}),\\
K_s &= K_{s-1} - \frac{\varepsilon_s}{\lambda_s}K_{s-1}y_sy_s^{\top}K_{s-1},
\end{aligned} \tag{49}
$$

where $M_0 = M$, $C_0 = C$ and $K_0 = K$, and θ_s and ε_s are given by

$$\theta_s = y_s^\top M_{s-1} y_s \quad \text{and} \quad \varepsilon_s = \frac{\lambda_s - \mu_s}{1 - \lambda_s \mu_s \theta_s}. \tag{50}$$

By doing so, the eigenvalues will be embedded one at a time. However, it is possible to assign several of them at a time as long as the mass and stiffness matrices remain positive definite.

The method proposed below delays the updating of the coefficient matrices until all the real numbers, $\{\theta_s\}$ and $\{\varepsilon_s\}$, needed for the multi-assignment, have been computed. After all these quantities have been computed, the coefficient matrices are updated with only one rank-m_r symmetric update. The process will not only be more efficient than that which assigns one eigenvalue at a time, but it will be rich in Basic Linear Algebra Subroutines Level 3 (BLAS-3), such as matrix–matrix multiplications, rank-r updates, etc., which will make it suitable for high-performance computing.

Given r real numbers, $\{\mu_1, \ldots, \mu_r\}$, the following method computes a positive integer, $m_r \leqslant r$, the matrices W and U, and the diagonal matrices D_M, D_C and D_K, such that the updated symmetric matrix pencil, $F_{\text{new}}(\lambda) = \lambda^2 M_{\text{new}} + \lambda C_{\text{new}} + K_{\text{new}}$, has the spectrum

$$\text{spec}(F_{\text{new}}(\lambda)) = \{\mu_1, \ldots, \mu_{m_r}, \lambda_{m_r+1}, \ldots, \lambda_{2n}\},$$

where

$$M_{\text{new}} = M - W D_M W^\top, \tag{51}$$

$$C_{\text{new}} = C + U D_C W^\top + W D_C U^\top, \tag{52}$$

$$K_{\text{new}} = K - U D_K U^\top. \tag{53}$$

To develop formula (51), we consider the m_rth iteration of Eq. (49) and observe that

$$\begin{aligned} M_{m_r} &= M_0 - \sum_{s=1}^{m_r} \varepsilon_s \lambda_s M_{s-1} y_s y_s^\top M_{s-1} \\ &= M_0 - [M_0 y_1, \ldots, M_{m_r-1} y_{m_r}] \begin{bmatrix} \varepsilon_1 \lambda_1 & & \\ & \ddots & \\ & & \varepsilon_{m_r} \lambda_{m_r} \end{bmatrix} [M_0 y_1, \ldots, M_{m_r-1} y_{m_r}]^\top. \end{aligned} \tag{54}$$

We also observe that, for $s = 1, \ldots, m_r$,

$$\begin{aligned} \theta_s &= y_s^\top M_{s-1} y_s \\ &= y_s^\top [M_{s-2} - \varepsilon_{s-1} \lambda_{s-1} M_{s-2} y_{s-1} y_{s-1}^\top M_{s-2}] y_s \\ &= y_s^\top M_{s-2} y_s - \varepsilon_{s-1} \lambda_{s-1} (y_s^\top M_{s-2} y_{s-1})(y_{s-1}^\top M_{s-2} y_s) \end{aligned} \tag{55}$$

and

$$\begin{aligned} M_{s-1} y_s &= [M_{s-2} - \varepsilon_{s-1} \lambda_{s-1} M_{s-2} y_{s-1} y_{s-1}^\top M_{s-2}] y_s \\ &= M_{s-2} y_s - \varepsilon_{s-1} \lambda_{s-1} (y_{s-1}^\top M_{s-2} y_s) M_{s-2} y_{s-1}. \end{aligned} \tag{56}$$

Therefore, formula (51) can be derived from Eq. (54) by letting

$$W = [M_0 y_1, \ldots, M_{m_r-1} y_{m_r}] \quad \text{and} \quad D_M = \begin{bmatrix} \varepsilon_1 \lambda_1 & & \\ & \ddots & \\ & & \varepsilon_{m_r} \lambda_{m_r} \end{bmatrix}.$$

In addition, the matrices D_M and W can be determined by using recursions (55) and (56).

Similarly, formulae (52) and (53) can be obtained for the appropriate matrices U, D_K and D_C. Our discussions above are summarized in the algorithms below.

Algorithm 5.1 (*Simultaneous Assignment of Real Eigenvalues*).
Input:

(i) A set of real numbers $\{\mu_i\}_{i=1}^{r}$,
(ii) A set of unwanted real eigenpairs $\{(\lambda_i, y_i)\}_{i=1}^{r}$,
(iii) Symmetric matrices M, C and K such that M, and K are positive definite.

Output: Integer m_r, and the symmetric matrices M_{new}, C_{new} and K_{new} such that the updated quadratic matrix pencil $F_{\text{new}}(\lambda)$ contains the m_r eigenvalues in the spectrum ($m_r \leqslant r$) while the other eigenvalues and the associated eigenvectors remain unchanged.

Step 1: Compute $m_i = My_i$, $k_i = Ky_i$, $i = 1, \ldots, r$.
Step 2: Compute $\alpha_{ij} = y_i^\top m_j$, $\beta_{ij} = y_i^\top k_j$, $j = i, \ldots, r$, $i = 1, \ldots, r$.
Step 3: Set $\eta_1 = \sqrt{y_1^\top K y_1}$
  Update $\alpha_{1j} = \alpha_{1j}/\eta_1$, $\beta_{1j} = \beta_{1j}/\eta_1$, $j = 1, \ldots, r$.
Step 4: Set $\varepsilon_1 = \frac{\lambda_1 - \mu_1}{1 - \lambda_1 \mu_1 \alpha_{11}}$.
Step 5: For $s = 2, \ldots, r$.
  For $i = s, \ldots, r$.
    For $j = i, \ldots, r$.
      Update $\alpha_{ij} = \alpha_{ij} - \varepsilon_{s-1}\lambda_{s-1}\alpha_{s-1,i}\alpha_{s-1,j}$, $\beta_{ij} = \beta_{ij} - \frac{\varepsilon_{s-1}}{\lambda_{s-1}}\beta_{s-1,i}\beta_{s-1,j}$.
    End for j.
  End for i.
  Compute $\varepsilon_s = \frac{\lambda_s - \mu_s}{1 - \lambda_s \mu_s \alpha_{ss}}$.
  If $\beta_{ss} > 0$,then
    Compute $\eta_s = \sqrt{\beta_{ss}}$.
    Update $\alpha_{i,s} = \alpha_{i,s}/\eta_s$, $\beta_{i,s} = \beta_{i,s}/\eta_s$, $i = 1, \ldots, s$.
    Update $\alpha_{s,j} = \alpha_{s,j}/\eta_s$, $\beta_{s,j} = \beta_{s,j}/\eta_s$, $j = 1, \ldots, s$.
    Compute $m_r = s$.
  Else Exit Loop.
End for s.
Step 6: Normalize $m_i = m_i/\eta_i$, $k_i = k_i/\eta_i$, $i = 1, \ldots, m_r$.

Table 1
Approximate flop counts for embedding r ($r \ll n$) real eigenvalues

Strategy	Parameters	M_{new}, C_{new}, K_{new}	Total
r sequential assignment	$6n^2r$	$\frac{7n^2r}{2}$	$\frac{19n^2r}{2}$
Simultaneous assignment	$4n^2r + 4nr^2$	$3n^2r + 3nr$	$7n^2r + 4nr^2$

Step 7: For $s = 2, \ldots, m_r$,
For $i = s, \ldots, m_r$,
Update $m_i = m_i - \varepsilon_{s-1}\lambda_{s-1}\alpha_{s-1,i}m_{s-1}$, $k_i = k_i - \frac{\varepsilon_{s-1}}{\lambda_{s-1}}\beta_{s-1,i}k_{s-1}$.
End for i.
End for s.

Step 8: Set $W = [m_1, m_2, \ldots, m_{m_r}]$, $U = [k_1, k_2, \ldots, k_{m_r}]$, $D_M = \text{diag}(\varepsilon_1\lambda_1, \varepsilon_2\lambda_2, \ldots, \varepsilon_{m_r}\lambda_{m_r})$,

$$D_C = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{m_r}) \quad \text{and} \quad D_K = \text{diag}\left(\frac{\varepsilon_1}{\lambda_1}, \frac{\varepsilon_2}{\lambda_2}, \ldots, \frac{\varepsilon_{m_r}}{\lambda_{m_r}}\right).$$

Step 9: Update

$$M_{\text{new}} = M - WD_M W^\top,$$
$$C_{\text{new}} = C + UD_C W^\top + WD_C U^\top,$$
$$K_{\text{new}} = K - UD_K U^\top.$$

Return

To show the efficiency of the simultaneous assignment process, we compare the *flop counts* of Algorithm 5.1, with those of the successive assignment strategy by using non-equivalence transformation (7). In Table 1, we list the *flop counts* of these two methods.

From Table 1, we see that the simultaneous assignment method is more efficient than the successive assignment procedure.

6. Numerical results

In this section, we illustrate the efficiency and reliability of the proposed method by using two examples: The first one is taken from Harwell–Boeing Collections [25]. The data of the second is the *simulated data of a real-life aerospace example* provided to us by the Boeing company.

All numerical implementations were performed on a IBM Pentium III machine using MATLAB.

6.1. Example 1 (Updating of a statistically condensed oil rig model)

Consider the model (M, D, K) where

- The matrices $M \in \mathbb{R}^{66\times 66}$ and $K \in \mathbb{R}^{66\times 66}$ come from the statically condensed oil rig model of the *Harwell–Boeing* set *BCSSTRUC1* [25]. The matrix M is symmetric positive definite and the matrix K is symmetric positive semi-definite.
- The damping matrix C is defined by $C = \rho I_{66}$, with $\rho = 1.55$.

This model has 132 eigenvalues out of which eight are real eigenvalues $\{\lambda_1, \ldots, \lambda_8\}$, given by

$$\{\lambda_1 \ \lambda_2 \ \lambda_3 \ \lambda_4\} = \{-3.4628 \ -3.5709 \ -5.3584 \ -9.2761\},$$
$$\{\lambda_5 \ \lambda_6 \ \lambda_7 \ \lambda_8\} = \{-13.1972 \ -13.4480 \ -27.5536 \ -44.5031\}$$

and 62 pairs of complex conjugate eigenvalues that are not shown here. The set $\{\lambda_1, \ldots, \lambda_8\}$ is changed to the set $\{\mu_1, \ldots, \mu_8\}$, where

$$\{\mu_1 \ \mu_2 \ \mu_3 \ \mu_4\} = \{-3.32 \ -3.75 \ -5.05 \ -9.07\},$$
$$\{\mu_5 \ \mu_6 \ \mu_7 \ \mu_8\} = \{-13.59 \ -13.04 \ -27.31 \ -42.11\}.$$

Algorithm 5.1 is then applied, giving matrices D_M, D_C, and D_K as follows:

$$D_M = \mathrm{diag}(0.6697 \ -0.9138 \ 3.6368 \ -2.4231 \ 2.6340 \ -2.6111 \ 17.3927 \ -197.1462),$$
$$D_C = \mathrm{diag}(-0.1934 \ 0.2559 \ -0.6787 \ 0.2612 \ -0.1996 \ 0.1942 \ -0.6312 \ 4.4299),$$
$$D_K = \mathrm{diag}(0.0558 \ -0.0717 \ 0.1267 \ -0.0282 \ 0.0151 \ -0.0144 \ 0.0229 \ -0.0995).$$

The matrices W and U are not shown here. The matrices $M_{\mathrm{new}}, C_{\mathrm{new}}$ and K_{new} are then computed, using the update formulas, as a single rank-8 update of the matrices M, D, and K.

Verification: Define

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{132}),$$
$$\tilde{\Lambda} = \mathrm{diag}(\mu_1, \ldots, \mu_8, \lambda_9, \ldots \lambda_{132}),$$
$$Y = [y_1 \ldots y_{132}]$$

then

$$\|M_{\mathrm{new}} Y \tilde{\Lambda}^2 + D_{\mathrm{new}} Y \tilde{\Lambda} + K_{\mathrm{new}} X\|_F = 1.7709 \times 10^{-7},$$

which shows that the multiple embedding was successful and produced no spill-over.

Fig. 5 shows the bar graphs of the magnitude of the components of the matrix $K - K_{\mathrm{new}}$. Similar graphs exist for the matrices $M - M_{\mathrm{new}}$ and $D - D_{\mathrm{new}}$.

6.2. Example 2

The Boeing Simulated Example. The test matrices K, C, M in this example come from an aerospace industry problem in constructing aircraft structural models.

Ten complex conjugate pairs of eigenvalues, which seem to be “troublesome”, need to be embedded in the given model. This is done by applying Algorithm 3.1 ten times, assigning one pair at a time. The results of implementation are plotted in the figure below. To understand the error behaviors more clearly, both the absolute and the logarithms of the error matrices have been computed and the absolute errors for the stiffness matrix are shown here in Fig. 6. The logarithm

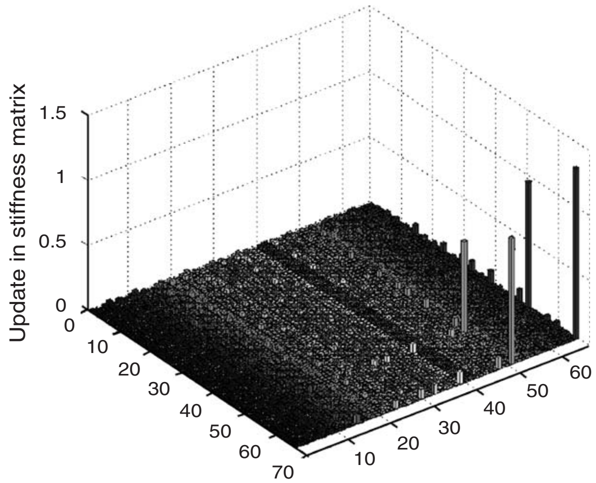

Fig. 5. Magnitudes of the entries of the matrix $K - K_{new}$.

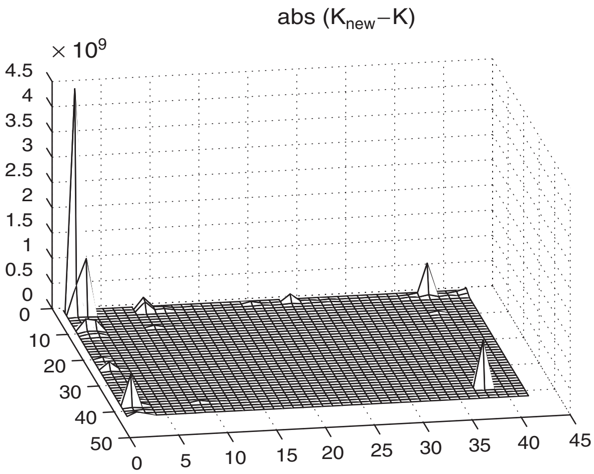

Fig. 6. The absolute error of $|K_{new} - K|$.

of the matrix M, denoted by $\log M$, is defined by

$$\log M(i,j) = \begin{cases} \log_{10}|M_{new}(i,j) - M(i,j)| & \text{if } |M_{new}(i,j) - M(i,j)| > 10^{-4}, \\ 0 & \text{otherwise.} \end{cases}$$

The results clearly show that our updating with low-rank transformations is successful. Furthermore, the results of the type obtained here provide an insight for the practicing engineers into what rows of the mass, stiffness or damping matrices need modification. *For this particular example*, *our plots show that the largest errors occur around* 3*rd and* 37*th rows and columns in all these matrices*. These rows and columns, therefore, need most modifications for the application under considerations.

7. Conclusion

The symmetric eigenvalue embedding problem addressed in this paper is the one of updating a symmetric finite element generated second-order model in such a way that the updated model remains symmetric, and a small subset of unwanted eigenvalues is replaced by a suitably user-chosen set, while the remaining large number of eigenvalues and eigenvectors do not change. The problem is intimately related to the partial eigenvalue assignment problem in control theory, which is usually solved by using feedback control. Unfortunately, with the use of feedback control, the symmetry of the model is completely destroyed. A novel symmetry preserving algorithm and the associated theories are presented in this paper. The proposed method results in a symmetric low-rank transformation of the original model, with the required properties. The method allows simultaneous assignment of several real eigenvalues; however, complex eigenvalues have to be assigned one at a time. Further research on simultaneous assignment of more than one complex eigenvalues is currently underaway. The results of the paper contribute to the progress in the solution of a well-known problem of immense practical importance in vibration industries: namely, the finite-element model updating problem, which is concerned with updating a symmetric finite-element model such that the updated model is symmetric, a small number of measured eigenvalues and eigenvectors from a practical structure is incorporated into the model, and the remaining large number of eigenvalues and eigenvectors that do not participate in the updating process remain invariant. Furthermore, because the proposed algorithms are rich in Basic Linear Algebra Subroutine-3 (BLAS-3) level operations, they can be implemented using high-performance software packages such as LAPACK on today's high-speed computers.

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